The Analysis Approach of Boundary Layer Equations of Power-Law Fluids of Second Grade

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A powerful analytic technique for nonlinear problems, the homotopy analysis method (HAM), is employed to give analytic solutions of power-law fluids of second grade. For the so-called second-order power-law fluids, the explicit analytic solutions are given by recursive formulas with constant coefficients. Also, for the real power-law index in a quite large range an analytic approach is proposed. It is demonstrated that the approximate solution agrees well with the finite difference solution. This provides further evidence that the homotopy analysis method is a powerful tool for finding excellent approximations to nonlinear equations of the power-law fluids of second grade.

Key words: Power-Law Fluid of Second Grade; Boundary Layers; Similarity Transformations; Homotopy Analysis Method; Series Solutions.

1. Introduction

Power-law fluids are by far the most widely used model to express non-Newtonian behaviour in fluids. The model predicts shear thinning and shear thickening behaviour. However, it has an inadequacy in expressing normal stress behaviour as observed in the die swelling and rod climbing behaviour in some non-Newtonian fluids. Normal stress effects can be expressed in the second grade fluid model, a special type of Rivlin-Ericksen fluids, but this model is incapable in representing shear thinning/thickening behaviour. A fluid model which exhibits all behaviours is deserved and Man and Sun [1, 2] proposed two models which they called “the power-law fluid of grade 2” and “modified second order (grade) fluid”. These models were slight modifications of a usual second grade fluid. The following power-law fluid of the second grade model is considered in this work:

$$T^* = -p^* I + \prod\nolimits^{m/2}(\mu A_1^* + \alpha_1 A_2^* + \alpha_2 A_1^{*2}), \quad (1)$$

where T^* is the Cauchy stress tensor, p^* is the pressure, I is the identity matrix, A_1^* and A_2^* are the first and second Rivlin-Ericksen tensors, respectively, μ, m, α_1 and α_2 are material moduli that may be constants or depend on temperature. For this model, when $m = 0$, $\alpha_1 = \alpha_2 = 0$, the fluid is Newtonian and hence μ represents the usual viscosity. $m = 0$ corresponds to the second grade fluid, $\alpha_1 = \alpha_2 = 0$ corresponds to the power-law fluid. The tensors are defined as

$$\begin{aligned} &A_1^* = L^* + L^{*T}, \quad A_2^* = \frac{\mathrm{d}A_1^*}{\mathrm{d}t^*} + A_1^* L^* + L^{*T} A_1^*, \\ &\prod = \frac{1}{2}\mathrm{tr}(A_1^{*2}), \quad L^* = \nabla^* v^*, \end{aligned} \quad (2)$$

where v^* is the velocity vector. The stars indicate that the quantities are dimensional. Model (1) satisfies the principle of material frame indifference.

Modifications of second grade fluids to account for shear thinning/thickening effects were considered recently in the literature. Man and Sun [1] first proposed the modifications. Later Man [2] considered the unsteady channel flow of a modified second grade fluid and existence, uniqueness and asymptotic stability of the solutions were exploited. Franchi and Straughan [3] presented a stability analysis of the modified model for a special viscosity function which depends linearly on the temperature. Gupta and Massoudi [4] investigated the flow of this fluid with temperature-dependent viscosity between heated plates. Massoudi and Phuoc [5] studied the flow down a heated inclined plane. The same authors [6] analyzed the pipe flow by Reynold’s temperature-dependent viscosity model. Hayat and Khan [7] studied the flow of the fluid over a porous flat plate and found solutions using the homotopy analysis method (HAM). Very recently, a symmetry analysis was presented for the boundary layer equations of the modified second grade fluid [8]. Detailed thermodynamic and stability analyses exist for second grade [9] and third grade [10] fluids. Dunn and Rajagopal [11] presented a critical review and thermodynamic analysis for fluids of differential type including the models considered here. Many issues regarding the applicability of such non-Newtonian models to real fluids, thermodynamic restrictions imposed on the constitutive equations and doubts raised in the previous literature on these models were addressed in detail.

In this paper, the boundary layer equations for model (1) are considered. Using a similarity transformation, the equations are reduced to an ordinary differential system. Numerical solutions of the ordinary differential equations are found using finite difference techniques. Numerical results are then compared with the approximate analytical results obtained by the HAM [12]. It is shown that HAM solutions agree well with the numerical solutions. The HAM has been applied successfully to many nonlinear problems in engineering and science, such the generalized Hirota-Satsuma coupled Korteweg-de Vries (KdV) equation [13], heat radiation [14], finding solitary-wave solutions for the fifth-order KdV equation [15], finding the solutions of the generalized Benjamin-Bona-Mahony equation [16], finding the root of nonlinear equations [17], finding the solitary-wave solutions for the Fitzhugh-Nagumo equation [18], unsteady boundary-layer flows over a stretching flat plate [19], exponentially decaying boundary layers [20], a nonlinear model of combined convective and radiative cooling of a spherical body [21], and many other problems (see [22–31], for example).

2. Equations of Motion

Mass conservation and linear momentum equations are

$$\operatorname{div} v^* = 0, \tag{3}$$

$$\rho \frac{\mathrm{d}v^*}{\mathrm{d}t^*} = \operatorname{div} T^* + \rho b^*. \tag{4}$$

Inserting (1) and (2) into (3) and (4), neglecting body forces, and making the usual classical boundary layer assumptions, one finally has the following boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial Y} = 0, \tag{5}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial Y} &= U\frac{\mathrm{d}U}{\mathrm{d}x} \\ &+ (m+1)\left(\frac{\partial u}{\partial Y}\right)^m \frac{\partial^2 u}{\partial Y^2} + k_1\left(\frac{\partial u}{\partial Y}\right)^{m-1} \\ &\cdot \left\{ m\left[\frac{\partial^2 u}{\partial Y^2}\left(u\frac{\partial^2 u}{\partial x \partial Y} + v\frac{\partial^2 u}{\partial Y^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial Y}\right)\right.\right. \\ &\left. - 2\left(\frac{\partial u}{\partial Y}\right)^2 \frac{\partial^2 u}{\partial x \partial Y}\right] + \frac{\partial u}{\partial Y}\left(v\frac{\partial^3 u}{\partial Y^3}\right. \\ &\left.\left. + u\frac{\partial^3 u}{\partial x \partial Y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial Y^2} - \frac{\partial u}{\partial Y}\frac{\partial^2 u}{\partial x \partial Y}\right)\right\}. \end{aligned} \tag{6}$$

The boundary conditions for the problem are

$$\begin{aligned} &u(x,0) = 0, \quad v(x,0) = 0, \quad u(x,\infty) = U(x), \\ &\frac{\partial u}{\partial Y}(x,\infty) = 0. \end{aligned} \tag{7}$$

For $m = 0$, the equations represent the boundary layers of a standard second grade fluid, and for $k_1 = 0$ the equations represent the boundary layers of a power-law fluid. From a scaling symmetry of the equations, the following similarity variable and functions are derived:

$$\begin{aligned} &\xi = \frac{Y}{x^{\frac{m}{m+2}}}, \quad u = x f(\xi), \quad v = x^{\frac{m}{m+2}} g(\xi), \\ &U = x. \end{aligned} \tag{8}$$

Substituting the similarity variables into the boundary layer equations yields the ordinary differential equation system

$$f - \frac{m}{m+2}\xi f' + g' = 0, \tag{9}$$

$$\begin{aligned} &f^2 - \frac{m}{m+2}\xi f f' + g f' = \\ &1 + (1+m)|f'|^{m-1} f' f'' + k_1 |f'|^{m-1}[2(m+1) f f' f'' \\ &- \frac{m^2}{m+2}\xi f f''^2 + m g f''^2 - 2\frac{2m+1}{m+2} f'^3 \\ &+ g f' f''' - \frac{m}{m+2}\xi f f' f'''], \end{aligned} \tag{10}$$

where the prime denotes differentiation with respect to the similarity variable ξ. The boundary conditions for the equations are

$$g(0)=0, \quad f(0)=0, \\ f(\infty)=1, \quad f'(\infty)=0. \tag{11}$$

For simplicity we assume that $f' \geq 0$ and $k_1 = 1$. Also, in the rest of the work, we let $0 < m < 1$.

3. Analytic Solutions for the Real Power-Law Index by HAM

For the real power-law index $0 < m < 1$, $f(\xi)$ and $g(\xi)$ can be expressed by a set of base functions:

$$\{\xi^k \exp(-n\xi)\,|\,k \geq 0, \quad n \geq 0\} \tag{12}$$

in the form

$$f(\xi) = \sum_{k,n=0}^{+\infty} f_{k,n}\xi^k \exp(-n\xi), \\ g(\xi) = \sum_{k,n=0}^{+\infty} g_{k,n}\xi^k \exp(-n\xi), \tag{13}$$

where $f_{k,n}$ and $g_{k,n}$ are coefficients to be determined. This provides us with the so-called *rule of solution expression*. Then, from the above expression and the boundary conditions (11), it is straightforward to choose

$$f_0(\xi) = 1 - \exp(-\xi), \\ g_0(\xi) = 1 - \exp(-\xi) \tag{14}$$

as the initial guesses of $f(\xi)$ and $g(\xi)$. According to (13) and the governing equations (9) and (10), we choose a system of linear operators:

$$\mathcal{L}_1[\phi(\xi;p)] = \frac{\partial \phi(\xi;p)}{\partial \xi}, \\ \mathcal{L}_2[\phi(\xi;p)] = \frac{\partial^3 \phi(\xi;p)}{\partial \xi^3} - \frac{\partial \phi(\xi;p)}{\partial \xi},$$

with the properties

$$\mathcal{L}_1[C_1] = 0, \\ \mathcal{L}_2[C_2 + C_3\exp(-\xi) + C_4\exp(\xi)] = 0.$$

We define

$$m - 1 = \varepsilon,$$

where $|\varepsilon| < 1$ is a real number. From (9) and (10), we define a system of nonlinear operators:

$$\mathcal{N}_1[\varphi(\xi;p), \psi(\xi;p)] = \varphi(\xi;p) - \frac{m}{m+2}\xi\varphi'(\xi;p) \\ + \psi'(\xi;p), \tag{15}$$

$$\begin{aligned}
&\mathcal{N}_2[\varphi(\xi;p), \psi(\xi;p)] = \varphi^2(\xi;p) \\
&-\frac{m}{m+2}\xi\varphi(\xi;p)\varphi'(\xi;p) + \psi(\xi;p)\varphi'(\xi;p) \\
&-\Big\{1 + (1+m)(\varphi'(\xi;p))^{\varepsilon p}\varphi'(\xi;p)\varphi''(\xi;p) \\
&+(\varphi'(\xi;p))^{\varepsilon p}\Big[2(m+1)\varphi(\xi;p)\varphi'(\xi;p)\varphi''(\xi;p) \\
&-\frac{m^2}{m+2}\xi\varphi(\xi;p)(\varphi''(\xi;p))^2 + m\psi(\xi;p)(\varphi''(\xi;p))^2 \\
&-2\frac{2m+1}{m+2}\varphi'(\xi;p) + \psi(\xi;p)\varphi'(\xi;p)\varphi'''(\xi;p) \\
&-\frac{m}{m+2}\xi\varphi(\xi;p)\varphi'(\xi;p)\varphi'''(\xi;p)\Big]\Big\},
\end{aligned} \tag{16}$$

where the prime denotes differentiation with respect to the similarity variable ξ, like in [31].

Using the above definitions, we construct the *zero-order deformation equations*

$$(1-p)\mathcal{L}_1[\psi(\xi;p) - g_0(\xi)] = p\hbar_1\mathcal{N}_1[\varphi(\xi;p), \psi(\xi;p)], \\ (1-p)\mathcal{L}_2[\varphi(\xi;p) - f_0(\xi)] = p\hbar_2\mathcal{N}_2[\varphi(\xi;p), \psi(\xi;p)], \tag{17}$$

subject to the boundary conditions

$$\psi(0;p) = 0, \quad \varphi(0;p) = 0, \quad \varphi(\infty;p) = 1, \\ \frac{\partial\varphi(\xi;p)}{\partial\xi}\Big|_{\xi=+\infty} = 0. \tag{18}$$

Obviously, when $p = 0$ and $p = 1$, then

$$\varphi(\xi;0) = f_0(\xi), \quad \varphi(\xi;1) = f(\xi), \\ \psi(\xi;0) = g_0(\xi), \quad \psi(\xi;1) = g(\xi). \tag{19}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\varphi(\xi;p)$ varies from the initial guess $f_0(\xi)$ to the solution $f(\xi)$, and $\psi(\xi;p)$ varies from the initial guess $g_0(\xi)$ to the solution $g(\xi)$. Expanding $\varphi(\xi;p)$ and $\psi(\xi;p)$ in a Taylor series with respect to p, one has

$$\varphi(\xi;p) = f_0(\xi) + \sum_{n=1}^{+\infty} f_n(\xi)p^n, \\ \psi(\xi;p) = g_0(\xi) + \sum_{n=1}^{+\infty} g_n(\xi)p^n,$$

where

$$f_n(\xi) = \frac{1}{n!}\frac{\partial^n \varphi(\xi;p)}{\partial p^n}\Big|_{p=0},\quad g_n(\xi) = \frac{1}{n!}\frac{\partial^n \psi(\xi;p)}{\partial p^n}\Big|_{p=0}. \tag{20}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameters $\hbar_1$ and $\hbar_2$ are properly chosen, the above series converge at $p=1$, and one has

$$f(\xi) = f_0(\xi) + \sum_{n=1}^{+\infty} f_n(\xi),\quad g(\xi) = g_0(\xi) + \sum_{n=1}^{+\infty} g_n(\xi), \tag{21}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [12].

Differentiating the zero-order deformation equations (17) and (19) n times with respect to p and then dividing them by $n!$ and finally setting $p=0$, we have, due to the definitions (20), the *nth-order deformation equations*

$$\mathcal{L}_1[g_n(\xi) - \chi_n g_{n-1}(\xi)] = \hbar_1 R_{1,n}(\xi),\quad \mathcal{L}_2[f_n(\xi) - \chi_n f_{n-1}(\xi)] = \hbar_2 R_{2,n}(\xi), \tag{22}$$

subject to the boundary conditions

$$g_n(0)=0,\quad f_n(0)=0,\quad f_n(+\infty)=0,\quad f_n'(+\infty)=0, \tag{23}$$

where

$$R_{i,n}(\xi) = \frac{1}{(n-1)!}\frac{\partial^{n-1}\mathcal{N}_i[\varphi(\xi;p),\psi(\xi;p)]}{\partial p^{n-1}}\Big|_{p=0} \tag{24}$$

for $i=1,2$ and

$$\chi_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1. \end{cases}$$

It can be found that

$$R_{1,n}(\xi) = f_{n-1}(\xi) - \frac{m}{m+2}\xi f'n - 1(\xi) + g_{n-1}'(\xi)$$

and

$$\begin{aligned} R_{2,n}(\xi) = \sum_{k=0}^{n-1}\Big(& f_k(\xi) f_{n-k-1}(\xi) \\ & - \frac{m}{m+2}\xi f_k(\xi) f_{n-k-1}'(\xi) + g_k(\xi) f_{n-k-1}'(\xi)\Big) \\ & - (1-\chi_n) - \sum_{k=0}^{n-1} w_k(\xi) z_{n-k-1}(\xi), \end{aligned}$$

where

$$\begin{aligned} z_k(\xi) = {} & (1+m)\sum_{i=0}^{k-1} f_i'(\xi) f_{k-i-1}''(\xi) \\ & + \sum_{i=0}^{k-1}\sum_{j=0}^{i}\Big(2(m+1) f_j(\xi) f_{i-j}'(\xi) f_{k-i-1}''(\xi) \\ & - \frac{m^2}{m+2}\xi f_j(\xi) f_{i-j}''(\xi) f_{k-i-1}''(\xi) \\ & + m g_j(\xi) f_{i-j}''(\xi) f_{k-i-1}''(\xi) \\ & - 2\frac{2m+1}{m+2} f_j'(\xi) f_{i-j}'(\xi) f_{k-i-1}'(\xi) \\ & + g_j(\xi) f_{i-j}'(\xi) f_{k-i-1}'''(\xi) \\ & - \frac{m}{m+2}\xi f_j(\xi) f_{i-j}'(\xi) f_{k-i-1}'''(\xi)\Big) \end{aligned}$$

and

$$w_k(\xi) = \frac{1}{k!}\left(\frac{\partial^k}{\partial p^k}\left[\frac{\partial \varphi(\xi;p)}{\partial \xi}\right]^{\varepsilon p}\right)\Big|_{p=0}.$$

It can be verified that

$$\begin{aligned} w_0(\xi) &= 1, \\ w_1(\xi) &= \varepsilon \ln(f_0'(\xi)), \\ w_2(\xi) &= \frac{\varepsilon f_1'(\xi)}{f_0'(\xi)} + \left(\frac{\varepsilon^2}{2}\right)\ln^2(f_0'(\xi)), \\ w_3(\xi) &= \varepsilon\left(\frac{f_2'(\xi)}{f_0'(\xi)} - \frac{f_1'^2(\xi)}{2 f_0'^2(\xi)}\right) \\ &\quad + \varepsilon^2 \frac{f_1'(\xi)}{f_0'(\xi)}\ln(f_0'(\xi)) + \frac{\varepsilon^3}{6}\ln^3(f_0'(\xi)), \end{aligned}$$

and so on, where $\ln(f_0'(\xi)) = -\xi$.

Now, the solution of the nth-order deformation equations (22) for $n \ge 1$ become

$$\begin{aligned} g_n(\xi) &= \chi_n g_{n-1}(\xi) + \int_0^{\xi} R_{1,n}(\xi)\mathrm{d}\xi + C_1, \\ f_n(\xi) &= \hat{f}_n(\xi) + C_2 + C_3 \mathrm{e}^{-\xi} + C_4 \mathrm{e}^{\xi}, \end{aligned}$$

where $\hat{f}_n(\xi)$ is a special solution of (22) and the coefficients C_1, C_2, C_3 and C_4 are determined by the boundary conditions (23).

According to the boundary conditions (23) and the rule of solution expression (13), we have $C_1 = C_4 = 0$. Moreover, the unknowns C_2 and C_3 are governed by

$$\hat{f}_n(0) + C_2 + C_3 = 0,$$

$$\hat{f}_n(+\infty) + C_2 = 0,$$

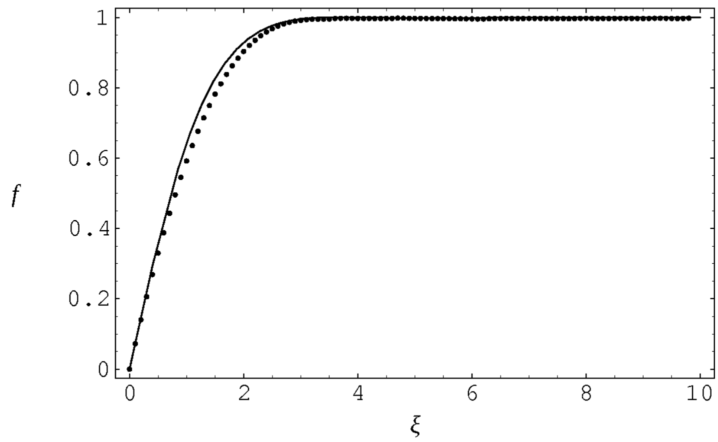

Fig. 1. Curve of the f function versus ξ for $m = 0.8$; solid curve, 10th-order approximation by the HAM; dotted curve, numerical solution.

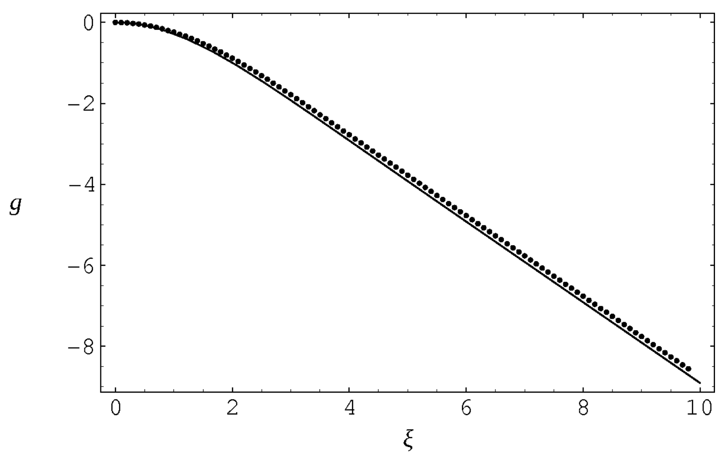

Fig. 2. Curve of the g function versus ξ for $m = 0.8$; solid curve, 10th-order approximation by the HAM; dotted curve, numerical solution.

which can be determined. Recall that $f_n'(+\infty) = 0$ is satisfied automatically.

At the Nth-order approximation, the analytic solutions are

$$f(\xi) \approx F_N(\xi) = \sum_{n=0}^{N} f_n(\xi),$$
$$g(\xi) \approx G_N(\xi) = \sum_{n=0}^{N} g_n(\xi).$$

The convergence of the solutions also depends on the choice of the auxiliary parameters $\hbar_1$ and $\hbar_2$.

4. Analysis and Results

Liao [12] proved that, as long as a series solution given by the HAM converges, it must be one of the exact solutions. So, it is important to ensure that the solution series (21) are convergent. In this paper, we seek the solutions with $f''(0) = 0$. By this condition, we can choose the proper values for $\hbar_1$ and $\hbar_2$. In what follows, we choose some values for m, and we compare the HAM results with the finite difference results.

Figures 1 and 2 show the HAM solutions obtained for $m = 0.8$. In this case we choose $\hbar_1 = -0.65957$ and $\hbar_2 = -0.17435$ with $F_{10}''(0) = 8.32667 \cdot 10^{-17}$.

Figures 3 and 4 show the HAM solutions obtained for $m = 0.4$. In this case we choose $\hbar_1 = -0.708756$ and $\hbar_2 = -0.20634$ with $F_{10}''(0) = -1.63064 \cdot 10^{-16}$.

Figures 5 and 6 show the HAM solutions obtained for $m = 0.2$. In this case we choose $\hbar_1 = \hbar_2 = -0.3096$ with $F_1''0(0) = -8.327 \cdot 10^{-17}$. But in this case the numerical solution obtained by finite difference method is not convergent, and we can not compare the HAM results.

5. Conclusions

In this paper, the homotopy analysis method (HAM) is employed to obtain the approximate analytic solu-

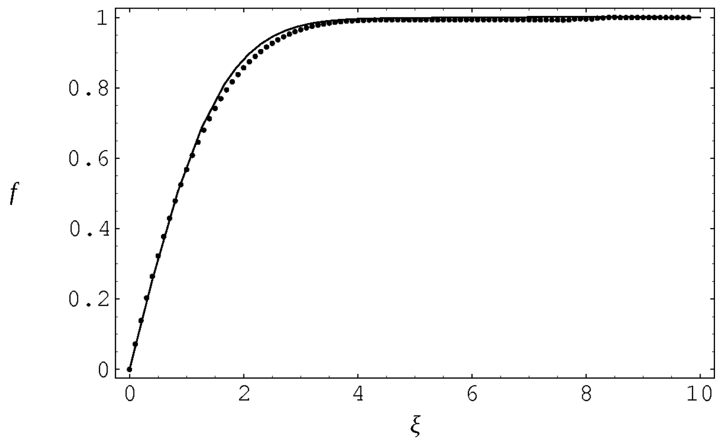

Fig. 3. Curve of the f function versus ξ for $m = 0.4$; solid curve, 10th-order approximation by the HAM; dotted curve, numerical solution.

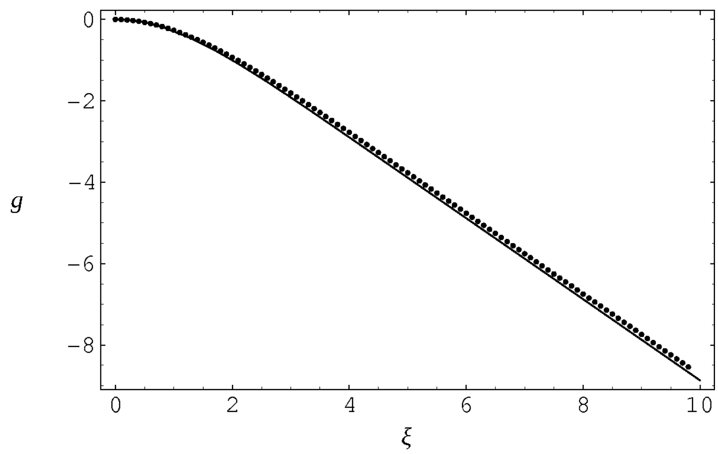

Fig. 4. Curve of the g function versus ξ for $m = 0.4$; solid curve, 10th-order approximation by the HAM; dotted curve, numerical solution.

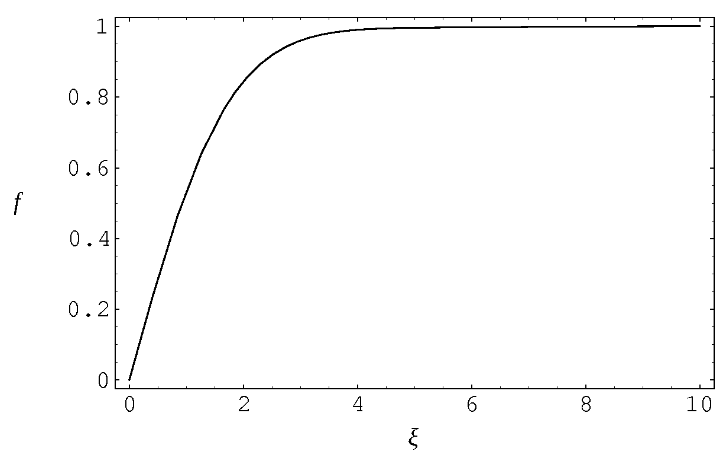

Fig. 5. Curve of the f function versus ξ for $m = 0.2$ for the 10th-order approximation by the HAM.

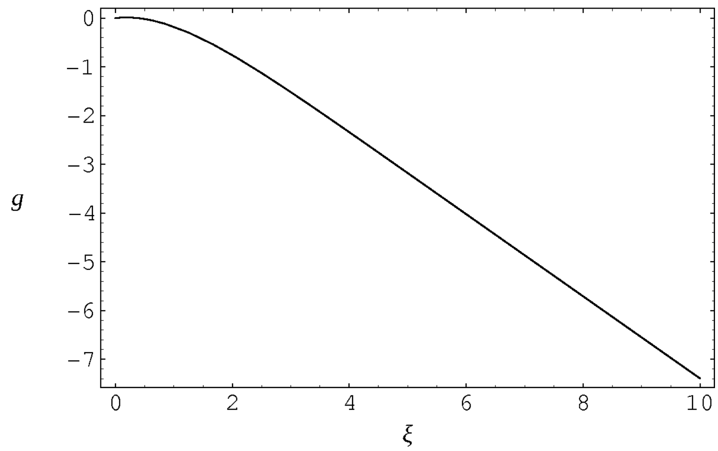

Fig. 6. Curve of the g function versus ξ for $m = 0.2$ for the 10th-order approximation by the HAM.

tions to the boundary layer flow of a power-law fluid of second grade. Finite difference solutions and HAM solutions are obtained for different power-law index m values. A good agreement is observed between the methods. HAM provides a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between the HAM and other methods. So, examples show the flexibility and potential of the HAM for complicated nonlinear problems in engineering.

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